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Synthetic Vector Analysis II

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Physicists prefer approximate calculations. This is natural, since physics is an empirical science. What is surprising judicious mathematicians is that many physicists use their favorite approximate reasoning to establish such theorems of pure mathematics as Gauss's divergence theorem and Stokes' theorem. What is more surprising is that their discussions impressively appeal to our geometric and physical intuitions, so that the discussions appear cryptically convincing, though mathematicians feel forced to contend offhand that such discussions are mathematically untenable and flimsy. In our previous paper (Nishimura, H. (2002). *International Journal of Theoretical Physics* **41**, 1165–1190) we have shown that once we realize that their discussions in establishing Gauss's divergence theorem and Stokes' theorem are not approximate (with errors) but infinitesimal (without errors), the discussions are bona fide authentic. What we should do is only transfer between the standard universe of sets and mappings whose set of real numbers contains no infinitesimals but zero and an intuitionistic universe of sets and mappings whose set of real numbers contains nilpotent infinitesimals in abundance and in coherence. The principal objective in this paper is to show that the same finesse can establish the celebrated Gauss–Bonnet theorem relating the topology and the Gaussian curvature of a surface, opening the way to the geometric theory of characteristic classes.

KEY WORDS: infinitesimal calculation; approximate calculation; nilpotent infinitesimal; synthetic differential geometry; vector analysis; divergence theorem; Stokes' theorem; Gauss–Bonnet theorem; Gaussian curvature.

1. INTRODUCTION

Physicists prefer approximate discussions. They are apt to identify a smooth function with its Taylor expansion up to a certain finite-order on the pretext that the quantities at issue are very small. Generally speaking, approximate discussions (with errors) yield approximate conclusions (with errors). Therefore, it is astonishing and even astounding fastidious mathematicians that many physicists are adamant enough to use their favorite approximate discussions so as to get theorems of pure mathematics such as Gauss's divergence theorem and Stokes' theorem. It has long been an enigma why approximate discussions (with errors) can yield exact conclusions (without errors).

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In our previous paper (Nishimura, 2002), we have succeeded in unveiling physicists' sacraments. What has appeared to be approximate calculations is no other than *infinitesimal* calculations. While approximate calculations are generally haunted by errors, infinitesimal calculations are genuinely exact calculations. The confusion between approximate and infinitesimal calculations has led to a mystery. It is true that the set of real numbers in our standard universe $\mathcal{S}$ of sets and mappings contains no infinitesimals except zero. However, there is an intuitionistic universe $\mathcal{G}$ of sets and mappings in which *nilpotent* infinitesimals are available in abundance and in coherence so that infinitesimal reasoning is admissible and even recommendable. This is the world of *synthetic differential geometry*, for which the reader is referred to Kock (1981), Lavendhomme (1996), and Moerdijk and Reyes (1991). Furthermore, we can transfer rather freely between $\mathcal{S}$ and $\mathcal{G}$ so that we may prove a theorem (e.g., Gauss's divergence theorem) in $\mathcal{G}$ to get the corresponding theorem in $\mathcal{S}$. Since $\mathcal{G}$ enjoys an infinitesimal horizon clearly distinguished from local and global ones, it is often a good strategy firstly to establish an infinitesimal version of the desired theorem in $\mathcal{G}$ and then to elevate it to a local one in $\mathcal{G}$. Once a local version of the theorem is established in $\mathcal{G}$, then it is to be transferred to the corresponding local theorem of $\mathcal{S}$ which is then to be elevated to a global theorem anyway in $\mathcal{S}$. In our previous paper, we have established Gauss's divergence theorem and Stokes' theorem on the infinitesimal horizon and have shown how to elevate them to local ones in $\mathcal{G}$. The local ones are concerned with squares (in case of Stokes' theorem) and cubes (in case of Gauss's divergence theorem). Since any smooth manifold is triangulable, and triangles and tetrahedrons are to be divided into squares and cubes, respectively, our global versions of Gauss's divergence theorem and Stokes' theorem follow readily by division of a given figure into squares or cubes. In this way, we can retain mathematical rigor while appealing to physical and geometric intuitions. In particular, we should note that **rot** and **div** are so defined that Stokes' theorem and Gauss's divergence theorem hold on the infinitesimal level. We have no other choice in defining operations **rot** and **div** if we want to see these famous theorems obtain on the infinitesimal level. In other words, these two theorems determine **rot** and **div** uniquely, so that these two theorems are tautological on the infinitesimal level.

One of the star attractions of classical differential geometry is undoubtedly the Gauss–Bonnet theorem relating the topology and the Gaussian curvature of a surface. The principal objective in this paper is to show that a similar synthetic argument can establish the theorem. Our infinitesimal formulation of the theorem reveals the geometric meaning of the Gaussian curvature and determines the Gaussian curvature uniquely, just as Gauss's divergence theorem and Stokes' theorem on the infinitesimal horizon determine **div** and **rot** uniquely. Our local version of the theorem is stated for oriented squares. Since any surface, whether it is orientable or not, can be divided into oriented squares, the global version of the Gauss–Bonnet theorem can be elicited from our local version. It is to our great surprise that Gauss's divergence theorem and Stokes' theorem on the one hand and the Gauss–Bonnet theorem on the other hand should be consanguineous, which might transcend the most daring physicist's imagination. This is why the Gauss–Bonnet theorem is unusually discussed under the rubric of vector analysis.

Moerdijk and Reyes (1991, Chapter 5, §5) have already addressed the Gauss–Bonnet theorem from a synthetic perch, establishing an infinitesimal version of the Gauss–Bonnet theorem and then eliciting a local version from it. What they have called the infinitesimal Gauss–Bonnet theorem is an infinitesimal version of what is usually called the fundamental theorem of connection and curvature in orthodox differential geometry. What they have called the local Gauss–Bonnet theorem appears somewhat deficient to orthodox differential geometers, for they have not taken leaps of angles into account. This means that their local Gauss–Bonnet theorem as such does not even imply that the sum of the three interior angles of a geodesic triangle on a surface is π plus the integral of the Gaussian curvature over the triangle (in particular, the sum of the three interior angles of a triangle in the Euclidean plane is π), which would give Gauss' famous Theorema Egregium as a direct corollary (cf. Spivak, 1999, II, p. 143). We do not know whether and how Moerdijk and Reyes (1991) would like to elevate their local Gauss–Bonnet theorem to the global one, but in an orthodox approach to the Gauss–Bonnet theorem, a surface whose boundary is of jumps of angles (say, polygons) is approximated by surfaces with smooth boundaries, for which the reader is referred to Spivak (1999, III, pp. 266–268). We would like to contend that such an ad hoc argument is unnecessary. What we should do is only to take jumps of angles into account on the infinitesimal level from scratch. After fixing our basic framework in Section 2, we prove the infinitesimal and local Gauss–Bonnet theorems in Section 3 and 4, respectively. We will work in $\mathcal{G}$ throughout Sections 2–4. Concluding remarks concerning the elementary calculation of the sum of the three interior angles of a triangle in a Euclidean plane in our Gauss–Bonnet context and a topic of future study are given in Section 5.

2. THE BASIC FRAMEWORK

We assume that the reader is well familiar with rudiments of synthetic differential geometry such as seen in Lavendhomme (1996) up to Chapter 5. Our set $\mathbb{R}$ of real numbers is replete with nilpotent infinitesimals, abiding by the (general) Kock–Lawvere axiom. There is a relation $\leq$ on $\mathbb{R}$, which is a substitute for the total order of real numbers in orthodox mathematics. Note that $\leq$ is neither total nor partial. It is only a preorder. We have $0 \leq d$ and $d \leq 0$ for any $d \in D$, where D is the set of elements of $\mathbb{R}$ whose squares vanish. In our synthetic context the interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ does not determine its endpoints uniquely, so that, formally speaking, we should distinguish strictly between the interval $[a, b]$ and the marked interval $([a, b], a, b)$, both of which are loosely denoted by the same symbol $[a, b]$ in this paper. We write $a < b$ for $a \leq b$ and $a \neq b$. We say that a is *positive* if $0 < a$. We denote by $\mathbf{S}^1$ the set $\{(a, b) \in R^2 | a^2 + b^2 = 1\}$. The set $\mathbf{S}^1$ can naturally be identified with the $\mathbf{SO}(2, \mathbb{R})$ of matrices

$$\begin{pmatrix} a & -b \\ b & 1 \end{pmatrix}$$

with $a^2 + b^2 = 1$. Since the latter has a natural group structure, the former can also be regarded as a group. We assume that the group $\mathbf{S}^1$ can be identified with the group $\mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. We write v for the canonical projection $\mathbb{R} \to \mathbf{S}^1$. We require $v(0) = (1, 0)$ and $v(\pi) = (-1, 0)$. We assume that $v'(0) = (0, 1)$, where v is regarded as a mapping $\mathbb{R} \to \mathbb{R}^2$. Microlinear spaces play the same role in synthetic differential geometry as smooth manifolds have played in orthodox differential geometry. They are characterized as spaces which enjoy a certain transfer principle from $\mathbb{R}$ on the infinitesimal level, just as smooth manifolds are characterized by means of local charts on the local level in orthodox differential geometry. An arbitrarily chosen microlinear space M shall be fixed throughout the paper. Given $x \in M$, we denote by $\mathbf{T}_x(M)$ the totality of tangent vectors to M at x, while we denote by $\mathbf{T}(M)$ the totality of tangent vectors to M, and the canonical mapping $\mathbf{T}(M) \to M$ assigning $t(0)$ to each $t : D \to M$ is denoted by τ_M. Two nonzero tangent vectors t_1, t_2 to M at the same point are considered to be *equivalent* if there exists a positive element a of $\mathbb{R}$ with $t_1 = at_2$. The resulting equivalence classes are called rays, and their totality is denoted by $\mathrm{Rays}(M)$. The ray determined by a nonzero tangent vector t to M is denoted by $\tilde{t}$. By assigning $t(0)$ to the ray $\tilde{t}$ represented by a nonzero tangent vector t to M, we have the canonical mapping $t_M : \mathrm{Rays}(M) \to M$. For each $x \in M$ we write $\mathrm{Rays}_x(M)$ for the totality of rays to M that are mapped by t_M to x. It is assumed that $\mathrm{Rays}_x(M)$ is nonempty for each $x \in M$. We assume that $\mathbf{S}^1$ acts freely and transitively on $\mathrm{Rays}_x(M)$ to the right for each $x \in M$, so that the canonical mapping $t_M : \mathrm{Rays}(M) \to M$ is an $\mathbf{S}^1$-*bundle*. We require that $\tilde{t}(-1, 0) = (-\tilde{t})$ for any nonzero tangent vector t to M.

Given two nonzero tangent vectors t_1, t_2 to M at the same point, an element $a \in \mathbb{R}$ with $\tilde{t}_1 v(a) = \tilde{t}_2$ is called the *angle* from t_1 to t_2 and is denoted by $\measuredangle(t_1, t_2)$ or by $\measuredangle(\tilde{t}_1, \tilde{t}_2)$. Note that the angle $\measuredangle(t_1, t_2)$ is determined up to mod 2π. We will make it a rule to choose the angle $\measuredangle(t_1, t_2)$ with $-\pi < \measuredangle(t_1, t_2) < \pi$, as far as it is possible.

Lemma. 2.1. *Let t_1, t_2, t_3 be nonzero tangent vectors to M at the same point. Then we have*

$$\measuredangle(t_1, t_2) + \measuredangle(t_2, t_3) = \measuredangle(t_1, t_3) \pmod{2\pi} \tag{2.1}$$

$$\measuredangle(t_1, t_2) = -\measuredangle(t_2, t_1) \pmod{2\pi} \tag{2.2}$$

$$\measuredangle(-t_1, t_2) = \pi - \measuredangle(t_2, t_1) \pmod{2\pi} \tag{2.3}$$

$$\measuredangle(t_1, -t_2) = \pi - \measuredangle(t_2, t_1) \pmod{2\pi} \tag{2.4}$$

$$\measuredangle(-t_1, -t_2) = \measuredangle(t_1, t_2) \pmod{2\pi} \tag{2.5}$$

Proof: (2.1) and (2.2) should be obvious. Since $\measuredangle(-t_1, t_2) + \measuredangle(t_2, t_1) = \measuredangle(-t_1, t_1) = \pi \pmod{2\pi}$ by (2.1), (2.3) follows. (2.4) can be established similarly. For (2.5) we note that

$$\begin{aligned}
&\measuredangle(-t_1, -t_2) \\
&= \pi - \measuredangle(-t_2, t_1) \text{ [by (2.3)]} \\
&= \pi - \{\pi - \measuredangle(t_1, t_2)\} \text{ [by (2.4)]} \\
&= \measuredangle(t_1, t_2) \pmod{2\pi}
\end{aligned} \tag{2.6}$$

□

We should say that, as far as the rest of the paper is concerned, we are fortunate to see the above equalities hold absolutely (i.e., without respect to mod 2π). By way of example, if $-\pi < \measuredangle(t_1, t_2) < \pi$ and $\measuredangle(t_2, t_3) \in D$ in (1.1), then we can and should choose $\measuredangle(t_1, t_3)$ with $-\pi < \measuredangle(t_1, t_3) < \pi$, and (1.1) holds absolutely. Such easy comments will be omitted, and we will bluntly write the equalities absolutely throughout the rest of the paper.

A microsquare $\gamma : D^2 \to M$ is said to be *positively oriented* if $0 < \measuredangle(s_{0,0}, t_{0,0}) < \pi$ with $s_{0,e} = \gamma(., e)$ and $t_{e,0} = \gamma(e, .)$ being nonzero tangent vectors to M for any $e \in D$. A mapping $\gamma : [0, 1] \times D \to M$ is said to be *positively oriented* if the microsquare $(d_1, d_2) \in D^2 \mapsto \gamma(a + d_1, d_2)$ is positively oriented for any $a \in [0, 1]$. A mapping $\gamma : [0, 1] \times [0, 1] \to M$ is said to be *positively oriented* if the microsquare $(d_1, d_2) \in D^2 \mapsto \gamma(a + d_1, b + d_2)$ is positively oriented for any $a, b \in [0, 1]$.

An $\mathbf{S}^1$-*connection* on M is a mapping ∇ from $\mathrm{Rays}(M) \times_M \mathbf{T}(M) = \{(t, t) \in \mathrm{Rays}(M) \times \mathbf{T}(M) | t_M(t) = \tau_M(t)\}$ to $\mathrm{Rays}(M)^D$ abiding by the following conditions:

$$l_M \circ \nabla(t, t) = t \tag{2.7}$$

$$\nabla(t, t)(0) = t \tag{2.8}$$

$$\nabla(t, at)(d) = \nabla(t, t)(ad) \text{ for any } a \in \mathbb{R} \text{ and any } d \in D. \tag{2.9}$$

$$\nabla(t\xi, t)(d) = \nabla(t, t)(d)\xi \text{ for any } \xi \in \mathbf{S}^1 \text{ and any } d \in D. \tag{2.10}$$

$$\begin{aligned}
&\text{The mapping } \nabla(., t)(d) : \mathrm{Rays}_{t(0)}(M) \\
&\quad \to \mathrm{Rays}_{t(d)}(M) \text{ is bijective for any} d \in D.
\end{aligned} \tag{2.11}$$

The mapping in (2.11) is denoted by $p_{(t,d)}$, called the *parallel transport* along t from $t(0)$ to $t(d)$, while its inverse is denoted by $q_{(t,d)}$, called the *parallel transport* along t from $t(d)$ to $t(0)$. In the rest of the paper, given $\tilde{t}' \in$

$\mathrm{Rays}_{t(0)}(M)$ and $\tilde{t}'' \in \mathrm{Rays}_{t(d)}(M)$, we will loosely denote by $p_{(t,d)}(t')$ and $q_{(t,d)}(t'')$ nonzero tangent vectors to M representing $p_{(t,d)}(\tilde{t}')$ and $q_{(t,d)}(\tilde{t}'')$, respectively. An arbitrarily chosen $\mathbf{S}^1$-*connection* ∇ on M shall be fixed for the rest of the paper.

The following easy lemma is implicit in the succeeding two sections.

Lemma. 2.2. *For any positively oriented microsquare γ on M and any $d, e \in D$, we have*

$$0 < \measuredangle\big(p_{(s_{0,d,e})}(s_{0,d}),\ p_{(t_{e,0,d})}(t_{e,0})\big) < \pi \tag{2.12}$$

For any $t \in \mathrm{Rays}(M)^D$ there exists a unique $b \in \mathbb{R}$ such that

$$\measuredangle\big(p_{(t_{M^\circ t,d})}(t(0)),\ t(d)\big) = bd \tag{2.13}$$

for any $d \in D$. We have

Proposition 2.3. *By assigning the above b to $t \in \mathrm{Rays}(M)^D$, we have a (singular) differential 1-form on* Rays*(M), which is denoted by ω and is called the connection form.* □

The following theorem is an infinitesimal version of what is usually called the fundamental theorem of connection and curvature in orthodox differential geometry. Its proof is based on the infinitesimal version of Stokes' theorem.

Theorem 2.4. *For any $\gamma \in \mathrm{Rays}(M)^{D^2}$ and any $e_1, e_2 \in D$, we have*

$$\mathbf{d}\omega(\gamma)e_1e_2 = \measuredangle\big(q_{(t_{0,0},e_2)}{}^{\circ}q_{(s_{0,e_2},e_1)}{}^{\circ}\,p_{(t_{s_1,0},e_2)}{}^{\circ}\,p_{(s_{0,0},e_1)}(\gamma(0,0)),\ \gamma(0,0)\big), \tag{2.14}$$

where $\mathbf{d}\omega$ is the exterior differential of ω, $s_{0,0} = \gamma(.,0)$; $s_{0,e_2} = \gamma(.,e_2)$; $t_{0,0} = \gamma(0,.)$; and $t_{e_1,0} = \gamma(e_1,.)$.

Proof: We have

$$\begin{aligned}
\mathbf{d}\omega(\gamma)e_1e_2 &= \omega(s_{0,0})e_1 + \omega\big(t_{e_1,0}\big)e_2 - \omega\big(s_{0,e_2}\big)e_1 - \omega(t_{0,0})e_2 \\
&= \measuredangle\big(p_{(s_{0,0},e_1)}(\gamma(0,0)),\ \gamma(e_1,0)\big) + \measuredangle\big(p_{(t_{e_1,0},e_2)}(\gamma(e_1,0)),\ \gamma(e_1,e_2)\big) \\
&\quad - \measuredangle\big(p_{(s_{0e_2},e_1)}(\gamma(0,e_2)),\ \gamma(e_1,e_2)\big) - \measuredangle\big(p_{(t_{0,0},e_2)}(\gamma(0,0)),\ \gamma(0,e_2)\big) \\
&= \measuredangle\big(p_{(t_{e_10},e_2)} \circ p_{(s_{0,0},e_1)}(\gamma(0,0)),\ p_{(t_{e_1,0},e_2)}(\gamma(e_1,0))\big) \\
&\quad + \measuredangle\big(p_{(t_{e_1,0},e_2)}(\gamma(e_1,0)),\ \gamma(e_1,e_2)\big) + \measuredangle\big(\gamma(e_1,e_2),\ p_{(s_{0,e_2},e_1)}(\gamma(0,e_2))\big) \\
&\quad + \measuredangle\big(p_{(s_{0,e_2},e_1)}\big(\gamma(0,e_2),\ p_{(s_{0,e_2},e_1)} \circ p_{(t_{0,0}e_2)}(\gamma(0,0))\big)
\end{aligned}$$

$$= \measuredangle\big(p_{(t_{e_q,0},e_2)} \circ p_{(s_{0,0},e_1)}(\gamma(0,0)),\ p_{(s_{0,e_2},e_1)} \circ p_{(t_{0,0},e_2)}(\gamma(0,0))\big)$$

$$= \measuredangle\big(q_{(t_{0,0},e_2)} \circ q_{(s_{0,e_2},e_1)} \circ p_{(t_{e_1,0},e_2)} \circ p_{(s_{0,0},e_1)}(\gamma(0,0)),\ \gamma(0,0)\big) \tag{2.15}$$

□

Proposition 2.5. *There exists a (singular) differential 2-form* Ω *on* M *such that*

$$\mathbf{d}\omega(\gamma) = \Omega(t_M \circ \gamma) \tag{2.16}$$

for any $\gamma \in \mathrm{Rays}(M)^{D^2}$.

Proof: For any $\gamma \in M^{D^2}$ and any $t \in \mathrm{Rays}_{\gamma(0,0)}(M)$, we define $\varphi(\gamma, t) \in \mathrm{Rays}(M)^{D^2}$ to be

$$\varphi(\gamma, t)(e_1, e_2) = p_{(\gamma(e_1,.),e_2)} \circ p_{(\gamma(.,0),e_1)}(t) \tag{2.17}$$

for any $e_1, e_2 \in D$. It is easy to see that

$$\varphi(a \cdot_i \gamma, t) = a \cdot_i \varphi(\gamma, t) \tag{2.18}$$

for any $a \in \mathbb{R}$ and $i = 1, 2$. Since $\mathbf{d}\omega(\varphi(\gamma, t))$ does not depend on t by Theorem 1.4, we can define a (singular) differential 2-form Ω to be

$$\Omega(\gamma) = \mathbf{d}\omega(\varphi(\gamma, t)) \tag{2.19}$$

for any $t \in \mathrm{Rays}_{\gamma(0,0)}(M)$. It is easy to see that (2.16) holds. □

For any function $f : [0, 1] \to M$ with the tangent vector $f(. + a)$ being nonzero for each $a \in [0, 1]$, there exists a unique $b \in \mathbb{R}$ such that (2.20)

$$\measuredangle\big(p(_{f(.+a),d)}(f(. + a)),\ f(. + a + d)\big) = bd \tag{2.20}$$

for any $d \in D$. This b is called the *geodesic curvature* of f at a and is denoted by $\kappa_g(f, a)$.

Proposition 2.6. *For any function* $f : [0, 1] \to M$ *with the tangent vector* $f(. + a)$ *being nonzero for each* $a \in [0, 1]$, *we have*

$$\int_a^{a+d} \kappa_g(f, x)\, dx = \measuredangle\big(p_{(f(.+a),d)}(f(. + a)),\ f(. + a + d)\big)$$

for any $d \in D$.

Proof: This follows from Proposition 11 of Lavendhomme (1996, §1.3) and (2.20). □

3. THE INFINITESIMAL GAUSS–BONNET THEOREM

The following theorem is a microsquare version of the well-known classical theorem that the sum of the three interior angles of a geodesic triangle on a surface is π plus the integral of the Gaussian curvature over the triangle. We would like to call it the *infinitesimal Gauss–Bonnet theorem*.

Theorem 3.1. *Let γ be a positively oriented microsquare on M. Let $e_1, e_2 \in D$. Let $s_{0,e} = \gamma(., e)$ and $t_{e,0} = \gamma(e, .)$ for any $e \in D$. Then we have*

$$2\pi = \int_{[0,e_1]\times[0,e_2]} \Omega + \measuredangle\left(p_{(s_{0,0},e_1)}(s_{0,0}), t_{e_1,0}\right) + \measuredangle\left(p_{(t_{e_1,0,e_2})}\left(t_{e_1,0}\right),\right.$$
$$\left. - p_{(s_{0,e_2},e_1)}\left(s_{0,e_2}\right)\right) + \measuredangle\left(-s_{0,e_2}, -p_{(t_{0,0,e_2})}(t_{0,0})\right) + \measuredangle\left(-t_{0,0,S_{0,0}}\right) \tag{3.1}$$

Proof: It is easy to see that

$$\measuredangle\left(p_{(s_{0,0},e_1)}(S_{0,0}), t_{e_1,0}\right) = \measuredangle\left(s_{0,0}, q_{(s_{0,0},e_1)}\left(t_{e_1,0}\right)\right) \tag{3.2}$$

$$\measuredangle\left(p_{(t_{e_1,0,e_2})}\left(t_{e_1,0}\right), -p_{(s_{0,e_2,e_1})}\left(s_{0,e_2}\right)\right) \tag{3.3}$$
$$= \pi - \measuredangle\left(p_{(s_{0,e_2,e_1})}\left(S_{0,e_2}\right), p_{(t_{e_1},0,e_2)}\left(t_{e_1,0}\right)\right) \text{ [by (2.4)]}$$
$$= \pi - \measuredangle\left(q_{(t_{0,0},e_2)}\left(s_{0,e_2}\right), q_{(t_{0,0},e_2)} \circ q_{(s_{0},e_2,e_1)} \circ p_{(t_{e_1,0,e_2})}\left(t_{e_1,0}\right)\right)$$

$$\measuredangle\left(-s_{0,e_2}, -p_{(t_{0,0},e_2)}(t_{0,0})\right) \tag{3.4}$$
$$= \measuredangle\left(s_{0,e_2}, p_{(t_{0,0},e_2)}(t_{0,0})\right) \text{ [by (2.5)]}$$
$$= \measuredangle\left(q_{(t_{0,0},e_2)}\left(s_{0,e_2}\right), t_{0,0}\right)$$

$$\measuredangle(-t_{0,0}, s_{0,0}) \tag{3.5}$$
$$= \pi - \measuredangle(s_{0,0}, t_{0,0}) \text{ [by (2.3)]}$$

Therefore we have

$$\measuredangle\left(p_{(s_{0,0},e_1)}(s_{0,0}), t_{e_1,0}\right) + \measuredangle\left(p_{(t_{e_1},0,e_2)}\left(t_{e_1,0}\right), -p_{(s_0,e_2,e_1)}\left(s_{0,e_2}\right)\right)$$
$$+ \measuredangle\left(-s_{0,e_2}, -p_{(t_{0,0},e_2)}(t_{0,0})\right) + \measuredangle\left(-t_{0,0}, s_{0,0}\right)$$
$$= 2\pi + \left\{\measuredangle\left(s_{0,0}, q_{(s_{0,0},e_1)}\left(t_{e_1,0}\right)\right) - \measuredangle\left(s_{0,0}, t_{0,0}\right)\right\} + \left\{\measuredangle\left(q_{(t_{0,0},e_2)}\left(s_{0,e_2}\right), t_{0,0}\right)\right.$$
$$\left. -\measuredangle\left(q_{(t_{0,0},e_2)}\left(s_{0,e_2}\right), q_{(t_{0,0},e_2)} \circ q_{(s_0,e_2,e_1)} \circ p_{(t_{e_1},0,e_2)}\left(t_{e_1 0}\right)\right)\right\}$$
$$= 2\pi + \measuredangle\left(t_{0,0}, q_{(s_{0,0},e_1)}\left(t_{e_1 0}\right)\right) + \measuredangle\left(q_{(t_{0,0},e_2)} \circ q_{(s_0,e_2,e_1)} \circ p_{(t_{e_1},0,e_2)}\left(t_{e_1,0}\right), t_{0,0}\right)$$
$$\text{[by (1.1) and (1.2)]}$$
$$= 2\pi + \measuredangle\left(q_{(t_{0,0},e_2)} \circ q_{(s_0,e_2,e_1)} \circ p_{(t_{e_1},0,e_2)}\left(t_{e_1,0}\right), q_{(s_{0,0},e_1)}\left(t_{e_1,0}\right)\right) \text{ [by (2.1)]}$$

$$= 2\pi + \measuredangle\big(q_{(t_{0,0},e_2)} \circ q_{(s_0,e_2,e_1)} \circ p_{(t_{e_1,0,e_2})} \circ p_{(s_0,e_1,e_1)}\big(q_{(s_0,e_1,e_1)}\big(t_{e_1,0}\big)\big), q_{(s_{0,0},e_1)}\big(t_{e_1,0}\big)\big)$$
$$= 2\pi - \Omega(\gamma)e_1e_2, \tag{3.6}$$

which gives (2.1). □

4. THE LOCAL GAUSS–BONNET THEOREM

Let us begin with

Lemma 4.1. *Let $\gamma : [0, 1] \times D \to M$ be a positively oriented mapping. Let $e \in D$. Let $s_{x,d} = \gamma(. + x, d)$ and $t_{x,0} = \gamma(x, .)$ for any $x \in [0, 1]$ and any $d \in D$. Then we have*

$$\begin{aligned} 2\pi = &\int_{[0,1] \times [0,e]} \Omega + \measuredangle(s_{1,0}, t_{1,0}) + \measuredangle\big(p_{(t_{1,0},e)}(t_{1,0}), -s_{1,e}\big) \\ &+ \measuredangle\big(- s_{0,e}, -p_{(t_{0,0},e)}(t_{0,0})\big) + \measuredangle\big(- t_{0,0}, s_{0,0}\big) \\ &+ \int_1^0 \kappa_g(\gamma(., 0), x)\, dx - \int_0^1 \kappa_g(\gamma(., e), x)\, dx \end{aligned} \tag{4.1}$$

Proof: Let us define a function $f : [0, 1] \to \mathbb{R}$ as follows: For any $x \in [0, 1]$ we decree that

$$\begin{aligned} f(x) = &\int_{[0,x] \times [0,e]} \Omega + \measuredangle(s_{x,0}, t_{x,0}) + \measuredangle\big(p_{(t_{x,0},e)}(t_{x,0}), -s_{x,e}\big) \\ &+ \measuredangle\big(- s_{0,e}, -p_{(t_{0,0},e)}\big(t_{0,0}\big)\big) + \measuredangle\big(- t_{0,0}, s_{0,0}\big) \\ &+ \int_x^0 \kappa_g(\gamma(., 0), y)\, dy - \int_0^x \kappa_g(\gamma(., e), y)\, dy \end{aligned} \tag{4.2}$$

For each $d \in D$ we have

$$\begin{aligned} &f(x + d) - f(x) \\ = &\int_{[x,x+d] \times [0,e]} \Omega + \measuredangle(s_{x+d,0}, t_{x+d,0}) + \measuredangle\big(p_{(t_{x+d,0},e)}(t_{x+d,0}), -s_{x+d,e}\big) \\ &- \measuredangle\big(s_{x,0}, t_{x,0}\big) - \measuredangle\big(p_{(t_{z,0},e)}(t_{x,0}), -s_{x,e}\big) + \int_x^{x+d} \kappa_g(\lambda(., 0), y)\, dy \\ &- \int_x^{x+d} \kappa_g(\gamma(., g), y)\, dy \\ = &\int_{[x,x+d] \times [0,e]} \Omega + \measuredangle(s_{x+d,0}, t_{x+d,0}) + \measuredangle\big(p_{(t_{x+d,0},e)}(t_{x+d,0}), -s_{x+d,e}\big) \end{aligned}$$

$$
\begin{aligned}
& -\{\pi - \measuredangle(-t_{x,0}, s_{x,0})\} - \left\{\pi - \measuredangle\big(-s_{x,0}, -p_{(t_{x,0},e)}(t_{x,0})\big)\right\} \\
& + \int_x^{x+d} \kappa_g(\gamma(.,0), y)\,dy - \int_x^{x+d} \kappa g(\gamma(.,e), y)\,dy \text{ [by Lemma 2.1]} \\
= & -2\pi + \int_{[x,x+d]\times[0,e]} \Omega + \measuredangle(s_{x+d,0}, t_{x+d,0}) + \measuredangle\big(p_{(t_{x+d,0},e)}(t_{x+d,0}), \\
& - s_{x+d,e}\big) + \measuredangle(-t_{x,0}, s_{x,0}) + \measuredangle\big(-s_{x,0}, -p_{(t_x,0,e)}(t_{x,0})\big) \\
& + \measuredangle\big(p_{(s_{x,0},d)}(s_{x,0}), s_{x+d,0}\big) - \measuredangle\big(p_{(s_{x,e},d)}(s_{x,e}), s_{x+d,e}\big) \\
= & -2\pi + \int_{[x,x+d]\times[0,e]} \Omega + \measuredangle\big(p_{(s_{x,0},d)}(s_{x,0}), t_{x+d,0}\big) + \measuredangle\big(p_{(t_{x+d,0},e)}\big)(t_{x+d,0}), \\
& - p_{(s_{x,e,d})}(s_{x,e})) + \measuredangle\big(-t_{x,0}, s_{x,0}\big) + \measuredangle\big(-s_{x,0}, -p_{(t_{x,0},e)}(t_{x,0})\big) \\
= & \; 0 \text{ [by Theorem 3.1]}
\end{aligned} \tag{4.3}
$$

This means that $f' = 0$ on [0, 1], so that f is constant on [0, 1]. Since $f(0) = 2\pi$ trivially, we have $f = 2\pi$ on [0,1]. In particular, $f(1) = 2\pi$, which is tantamount to (4.1). □

Now we have the full version of the local Gauss–Bonnet theorem.

Theorem 4.2. *Let* $\gamma : [0,1] \times [0,1] \to M$ *be a positively oriented mapping. Let* $s_{a,b} = \gamma(. + a, b)$ *and* $t_{a,b} = \gamma(a, . + b)$ *for any* $a, b \in [0,1]$. *Then we have*

$$
\begin{aligned}
2\pi = & \int_{[0,1]\times[0,1]} \Omega + \measuredangle(s_{1,0}, t_{1,0}) + \measuredangle(t_{1,1}, -s_{1,1}) + \measuredangle(-s_{0,1}, -t_{0,1}) \\
& + \measuredangle(-t_{0,0}, s_{0,0}) + \int_0^1 \kappa_g(\gamma(.,0), x)\,dx + \int_0^1 \kappa_g(\gamma(1,.), x)\,dx \\
& - \int_0^1 \kappa_g(\gamma(.,1), x)\,dx - \int_0^1 \kappa_g(\gamma(0,.), x)\,dx
\end{aligned} \tag{4.4}
$$

Proof: Let us define a function $g : [0,1] \to \mathbb{R}$ as follows: For any $y \in [0,1]$ we decree that

$$
\begin{aligned}
g(y) = & \int_{[0,1]\times[0,y]} \Omega + \measuredangle(s_{1,0}, t_{1,0}) + \measuredangle(t_{1,y}, -s_{1,y}) + \measuredangle(-s_{0,y}, -t_{0,y}) \\
& + \measuredangle(-t_{0,0}, s_{0,0}) + \int_0^1 \kappa_g(\gamma(.,0), x)\,dx + \int_0^y \kappa_g(\gamma(1,.), x)\,dx \\
& - \int_0^1 \kappa_g(\gamma(.,1), y)\,dx - \int_0^y \kappa_g(\gamma(0,.), x)\,dx
\end{aligned} \tag{4.5}
$$

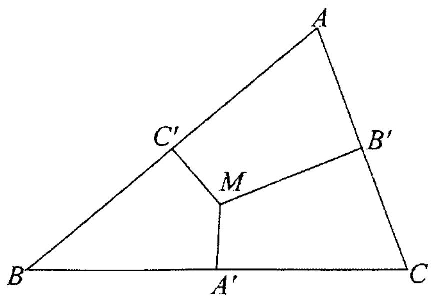
For each $d \in D$ we have

$$
\begin{aligned}
& g(y+d) - g(y) \\
= & \int_{[0,1]\times[y,y+d]} \Omega + \measuredangle(t_{1,y+d}, -s_{1,y+d}) + \measuredangle(-s_{0,y+d}, -t_{0,y+d}) - \measuredangle(t_{1,y}, -s_{1,y}) \\
& - \measuredangle(-s_{0,y}, -t_{0,y}) + \int_0^1 \kappa_g(\gamma(., y), x)\, dx + \int_y^{y+d} \kappa_g(\gamma(1, .), x)\, dx \\
& - \int_0^1 \kappa_g(\gamma(., y+d), x)\, dx - \int_y^{y+d} \kappa_g(\gamma(0, .), x)\, dx \\
= & \int_{[0,1]\times[y,y+d]} \Omega + \measuredangle(t_{1,y+d}, -s_{1,y+d}) + \measuredangle(-s_{0,y+d}, -t_{0,y+d}) \\
& - \{\pi - \measuredangle(s_{1,y}, t_{1,y})\} - \{\pi - \measuredangle(-t_{0,y}, s_{0,y})\} + \int_0^1 \kappa_g(\gamma(., y), x)\, dx \\
& + \int_y^{y+d} \kappa_g(\gamma(1, .), x)\, dx - \int_0^1 \kappa_g(\gamma(., y+d), x)\, dx \\
& - \int_y^{y+d} \kappa_g((0, .) \text{ [by Lemma 3.1]} \\
= & -2\pi + \int_{[0,1]\times[y,y+d]} \Omega + \measuredangle(t_{1,y+d}, -s_{1,y+d}) + \measuredangle(-s_{0,y+d}, -t_{0,y+d}) \\
& + \measuredangle(s_{1,y}, t_{1,y}) + \measuredangle(-t_{0,y}, s_{0,y}) + \int_0^1 \kappa_g(\gamma(., y), x)\, dx + \measuredangle\big(p_{(t_{1,y},d)}(t_{1,y}), t_{1,y+d}\big) \\
& - \int_0^1 \kappa_g(\gamma(., y+d), x)\, dx - \measuredangle\big(p_{(t_{0,y},d)}(t_{0,y}), s_{0,y+d}\big) \\
= & -2\pi + \int_{[0,1]\times[y,y+d]} \Omega + \measuredangle\big(p_{(t_{1,y},d)}(t_{1,y}), -s_{1,y+d}\big) \\
& + \measuredangle\big(-s_{0,y+d}, -p_{(t_{0,y},d)}(t_{0,y})\big) + \measuredangle(s_{1,y}, t_{1,y}) + \measuredangle(-t_{0,y}, s_{0,y}) \\
& + \int_0^1 \kappa_g(\gamma(., y), x)\, dx - \int_0^1 \kappa_g(\gamma(., y+d), x)\, dx \text{ [by Lemma 2.1]} \\
= & \; 0 \text{ [by Lemma 4.1]}
\end{aligned}
\tag{4.6}
$$

This means that $g' = 0$ on $[0, 1]$, so that g is constant on $[0, 1]$. Since $g(0) = 2\pi$ trivially, we have $g = 2\pi$ on $[0, 1]$. In particular, $g(1) = 2\pi$, which is tantamount to (4.4). □

5. CONCLUDING REMARKS

Once we establish the local version of the Gauss–Bonnet theorem concerning oriented squares, we transfer it from $\mathcal{G}$ to $\mathcal{S}$. Let us consider a triangle ABC in the Euclidean plane in $\mathcal{S}$. Let M be an interior point of the triangle, say, its mass center. Let A', B', and C' be the middle points of the edges BC, AC, and AB respectively.



By our local Gauss–Bonnet theorem applied to squares AC′ MB′, C′ BA′ M, and A′ CB′ M expressed in terms of interior angles in preference to exterior ones, we have

$$\measuredangle \mathrm{B'AC'} + \measuredangle \mathrm{AC'M} + \measuredangle \mathrm{C'MB'} + \measuredangle \mathrm{MB'A} = 2\pi \tag{5.1}$$

$$\measuredangle \mathrm{C'BA'} + \measuredangle \mathrm{BA'M} + \measuredangle \mathrm{A'MC'} + \measuredangle \mathrm{MC'B} = 2\pi \tag{5.2}$$

$$\measuredangle \mathrm{A'CB'} + \measuredangle \mathrm{CB'M} + \measuredangle \mathrm{B'MA'} + \measuredangle \mathrm{MA'C} = 2\pi \tag{5.3}$$

By adding the above three equations and noticing that

$$\measuredangle \mathrm{AC'M} + \measuredangle \mathrm{MC'B} = \pi \tag{5.4}$$

$$\measuredangle \mathrm{BA'M} + \measuredangle \mathrm{MA'C} = \pi \tag{5.5}$$

$$\measuredangle \mathrm{CB'M} + \measuredangle \mathrm{MB'A} = \pi \tag{5.6}$$

$$\measuredangle \mathrm{C'MB'} + \measuredangle \mathrm{A'MC'} + \measuredangle \mathrm{B'MA'} = 2\pi, \tag{5.7}$$

we have

$$\measuredangle \mathrm{CAB} + \measuredangle \mathrm{ABC} + \measuredangle \mathrm{BCA'} = \pi, \tag{5.8}$$

since $\measuredangle \mathrm{CAB} = \measuredangle \mathrm{B'AC'}$, $\measuredangle \mathrm{ABC} = \measuredangle \mathrm{C'BA'}$ and $\measuredangle \mathrm{BCA} = \measuredangle \mathrm{A'CB'}$ obviously. Thus, we have arrived at the familiar fact that the sum of the three interior angles of a triangle in a Euclidean plane is π.

Now we turn to an intriguing topic of future study. As is well known, the Gauss–Bonnet theorem has such higher-order generalizations as the Gauss–

Bonnet–Chern theorem and such far-reaching extensions as the Atiyah–Singer index theorem. We would be glad to see whether and how our synthetic approach to the Gauss–Bonnet theorem can be applied to such higher-dimensional or far-reaching generalizations.

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